

# **Minimal Coupling and Feynman's Proof**

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The non quantum relativistic version of the proof of Feynman for the Maxwell equations is discussed in a framework with a minimum number of hypotheses required. From the present point of view it is clear that the classical equations of motion corresponding to the gauge field interactions can be deduced from the minimal coupling rule, and we claim here resides the essence of the proof of Feynman.

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## **1. INTRODUCTION**

The proof of Feynman for the Maxwell equations presented in Dyson's paper (Dyson, 1990) was never published by Feynman himself because from his point of view the proof provides no new information about the classical or quantum nature of the electromagnetic field. Even though the proof is mathematically right, there are mixing physical inputs. First of all, the proof is based on the second law of Newton, which is a classical relation. Second, the quantum commutators between position and momentum are assumed (Dombey, 1991; Brehme, 1991; Anderson, 1991; Farquhar, 1991). Third, the framework is Galilean. Therefore, is quite surprising that with this information pure relativistic equations of motion emerge from this formalism. This is in fact the main result of the proof of Feynman. It is important to emphasize that the proof reproduces only the homogeneous Maxwell equations, which in fact are compatible with Galilean relativity (Vaidya and Farina, 1991).

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On the other hand, Dyson claims that the proof has a remarkable property in that it shows that the physics involved in the assumptions concerns only the homogeneous Maxwell equations. There are some results which show that Feynman's proof can be extended to the non-Abelian gauge fields (Lee, 1990) within a relativistic framework (Tanimura, 1992). Nevertheless, these last proposals have the same mixed physical inputs as the original proof (Farquhar, 1990). A straightforward extension was reviewed recently by Bracken (1998), who gave a derivation of the homogeneous Maxwell equations from a postulated set of Poisson brackets instead of the quantum commutators (just like Hughes, 1992). Bracken also proposed an extended formalism by postulating a set of relativistic Poisson brackets. Nevertheless this approach is, on one hand, non-manifestly covariant, and on the other hand, it is unable to derive the nonhomogeneous Maxwell equations, even though the field tensor may be built.

At this point, it seems that the most important physical property associated with the dynamics of a particle under the action of a gauge field is missed in all these approaches, the minimal coupling rule. It contains all the information of the sources and fields, and therefore it is a more natural starting point. In fact, it is equivalent to the assumptions involved in Feynman's proof, but it has the advantage of having a clear physical meaning; this is our main claim. Nothing of this seems to be new; however, this proof has attracted interest in the community because of its relationship to some fundamental aspects of physics. Thus our main motivation for giving the proof again is to illuminate its physical basis.

In keeping with this goal, our proof uses the minimum of hypotheses. It is based on the assumption that the minimal coupling rule holds. No quantum commutations relations are assumed. Section 2 reviews the original proof, Section 3 shows that Feynman's hypothesis can be obtained from the minimal coupling rule for a relativistic particle. So from the perspective of the present approach the validity of the minimal coupling rule is the essential element underlying Feynman's construction. Using the results of this section, we exhibit our approach explicitly in Sections 5 and 6, where the electromagnetic and the non-Abelian fields are considered, respectively.

## 2. FEYNMAN'S PROOF

We begin by reviewing the Feynman's proof. Essentially we follow the same approach as in Dyson (1990); our notations and conventions are the same. Let us consider a free particle with position and velocity  $x_i$  and  $\dot{x}_i = dx_i/dt$ , respectively. Then Newton's second law holds:

$$m\ddot{x}_j = F_j(x, \dot{x}, t) \quad (1)$$

Also the quantum commutation relations are assumed:

$$[x_j, x_k] = 0, \quad m[x_j, \dot{x}_k] = i\hbar\delta_{jk} \quad (2)$$

Then (1) and (2) imply

$$[x_j, F_k] + m[\dot{x}_j, \dot{x}_k] = 0 \quad (3)$$

Now because  $[x_j, F_k]$  is skew symmetric in the pair  $j$  and  $k$ , it allows us to introduce the auxiliary field  $H_l$  through

$$[x_j, F_k] = -\frac{i\hbar}{m} \epsilon_{jkl} H_l \quad (4)$$

and by using the Jacobi identity  $[x_l, [x_j, \dot{x}_k]] + \dot{x}_j, [\dot{x}_k, x_l] + [\dot{x}_k, [x_l, \dot{x}_j]] = 0$  together with (2) and (3), is straightforward to see  $H_l$  depends only on  $x$  and  $t$  because  $[x_l, [x_j, F_k]] = 0$ , or which is the same,

$$[x_l, H_m] = 0 \quad (5)$$

It is convenient to define a new field  $E_j$  by employing the relation

$$F_j = E_j + \epsilon_{jkl} \dot{x}_k H_l \quad (6)$$

which from (2), (4), and (5) satisfies  $[x_m, E_j] = 0$ , which means it does not depend on  $\dot{x}$ . On the other hand, by using (3) and (4), we can get for  $H_l$

$$H_l = \frac{m^2}{i2\hbar} \epsilon_{jkl} [\dot{x}_j, \dot{x}_k] \quad (7)$$

which together with the Jacobi identity allows us to obtain

$$\frac{\partial H_l}{\partial x_l} = [\dot{x}_l, H_l] = \frac{m^2}{i2\hbar} \epsilon_{jkl} [\dot{x}_l, [\dot{x}_k, \dot{x}_j]] = 0 \quad (8)$$

Next we take the total derivative of (7) with respect to  $t$ ,

$$\frac{\partial H_l}{\partial t} + \dot{x}_m \frac{\partial H_l}{\partial x_m} = \frac{m^2}{i\hbar} \epsilon_{jkl} [\ddot{x}_j, \dot{x}_k] \quad (9)$$

Finally, from (1) and (6) the RHS of (9) can be written as

$$\begin{aligned} & \frac{m}{i\hbar} \epsilon_{jkl} [E_j + \epsilon_{jmn} \dot{x}_m H_n, \dot{x}_k] \\ &= \frac{m}{i\hbar} (\epsilon_{jkl} [E_j, \dot{x}_k] + [\dot{x}_k H_l, \dot{x}_k] - [\dot{x}_l H_k, \dot{x}_k]) \\ &= \epsilon_{jkl} \frac{\partial E_j}{\partial x_k} + \dot{x}_k \frac{\partial H_l}{\partial x_k} + \dot{x}_l \frac{\partial H_k}{\partial x_k} + \frac{m}{i\hbar} H_k [\dot{x}_l, \dot{x}_k] \end{aligned} \quad (10)$$

In the last expression the third and fourth terms vanish because of (7) and (8). Therefore, by putting (10) in (9), we obtain Faraday's induction law,

$$\frac{\partial H_i}{\partial t} = \epsilon_{jkl} \frac{\partial E_j}{\partial x_k} \quad (11)$$

End of the proof.

Now, here it is important to make some remarks. First, note that the Galilean version of the Lorentz law, Eq. (6), has been explicitly used. Moreover, (2) means we are using a quantum framework. In other words, there are two mixed inputs: classical and quantum descriptions are combined in Feynman's proof. Even though these two classical and quantum aspects are taken into account, the result is amazing. The main result of the Feynman's proof is that, from quantum commutators [Eq. (2)] and the quantum version of the Newtonian force (Tanimura, 1992), the equations of motion for the fields are the homogeneous Maxwell equations. The proof can be extended to the case of non-Abelian gauge fields both in Newtonian (Lee, 1990) as well as in relativistic (Tanimura, 1992) dynamics. It is clear that because of the nature of the fields only the relativistic approach allows the construction of all equations of motion for the fields. Note also that Feynman's proof requires only the quantum commutation relations (Hughes, 1992), which for the purposes of the proof can be substituted by their classical version, the Poisson brackets. This key property raises the possibility of constructing a nonquantum version of the proof in the framework of special relativity with a minimum of hypotheses: the minimal coupling rule.

### 3. RELATIVISTIC CASE

Let us consider a relativistic particle with rest mass  $m$  in an inertial frame under the action of an external force in such a way that the generalized momentum satisfies the minimal coupling rule (see, for instance, O'Rai-fear-taigh, 1997)

$$\pi_\mu = m\dot{x}_\mu + A_\mu(x, \pi) \quad (12)$$

where  $\pi_\mu$  is the canonical momentum of the particle, which has the contribution of the fields through the potential  $A_\mu$ . In a general situation  $A_\mu$  might depend on the velocity of the particle or, which is the same, on the components of the canonical momentum. Consequently, the physical gauge fields will be deduced from particular restrictions on this dependence, as we shall show in the next sections. In fact, the proof we present below is totally general.

From now on, we will denote the derivative with respect to the proper time  $\tau$  as well as the derivatives with respect to the canonical coordinates of a phase space function  $f(x, \pi)$  by

$$\dot{f} \equiv \frac{df}{d\tau}, \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \tilde{\partial}^\mu \equiv \frac{\partial}{\partial \pi_\mu} \quad (13)$$

In this way, the Poisson bracket is given by

$$\{f, g\} \equiv \eta_{\rho\sigma} (\partial^\rho f \tilde{\partial}^\sigma g - \tilde{\partial}^\rho g \partial^\sigma f) \quad (14)$$

where  $\eta_{\rho\sigma}$  is the Minkowski metric.

Instead of taking Feynman's hypothesis, we are going to assume that the relation (12) holds. In other words, the relation (12) is put on a fundamental level, and using it, we shall show that the equations of motion for the fields, and the interaction law with a test particle, can be deduced without any additional assumption. So the present approach shows that the minimal coupling rule has all the dynamic information of the system.

From the definition of the Poisson brackets and the minimal coupling rule we get the relationship

$$m\{x_\mu, \dot{x}_\nu\} = \eta_{\mu\nu} - \tilde{\partial}_\mu A_\nu \quad (15)$$

which is the analog of (2) in the nonrelativistic case. Taking the derivative of (15) with respect to the proper time  $\tau$ , we have

$$m \frac{d}{d\tau} \{x_\mu, \dot{x}_\nu\} = m\{\dot{x}_\mu, \dot{x}_\nu\} + m\{x_\mu, \ddot{x}_\nu\} = -\frac{d}{d\tau} (\tilde{\partial}_\mu A_\nu) \quad (16)$$

Now, using the Jacobi identity for  $\{x_\nu, \{\dot{x}_\mu, \dot{x}_\rho\}\}$  and (15), one obtains

$$m\{x_\nu, \{\dot{x}_\mu, \dot{x}_\rho\}\} + \{\dot{x}_\mu, \tilde{\partial}_\nu A_\rho\} + \{\tilde{\partial}_\nu A_\mu, \dot{x}_\rho\} = 0 \quad (17)$$

which means that the quantity  $\{\dot{x}_\mu, \dot{x}_\rho\}$  depends on the derivative of  $x$  through the implicit dependence of  $A_\mu$  on  $\pi_\mu$ . This is the most general situation [compare with Eq. (5)]. Following Feynman, we define the skew-symmetric tensor

$$-\frac{1}{m} F_{\mu\nu} \equiv -m\{\dot{x}_\mu, \dot{x}_\nu\} = m\{x_\mu, \ddot{x}_\nu\} + \frac{d}{d\tau} (\tilde{\partial}_\mu A_\nu) \quad (18)$$

which, after it is expanded, takes the form of the tensor associated to the gauge field  $A_\mu$

$$F_{\mu\nu} = (\tilde{\partial}_\mu A_\nu - \tilde{\partial}_\nu A_\mu) + \{A_\mu, A_\nu\} \quad (19)$$

Notice that the last term of (19) must vanish for the electromagnetic case (see next section). In general it suggests the right form of the non-Abelian gauge field tensor. Taking the derivative with respect to  $x_\alpha$ , we obtain the following relation:

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = \partial_\alpha \{A_\mu, A_\nu\} + \partial_\mu \{A_\nu, A_\alpha\} + \partial_\nu \{A_\alpha, A_\mu\} \quad (20)$$

which suggests the definition of a ‘‘covariant derivative’’ of the form  $D_\alpha F_{\mu\nu} \equiv \partial_\alpha F_{\mu\nu} - \{F_{\mu\nu}, A_\alpha\}$ . This implies (20) can be written as

$$D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu} = 0 \quad (21)$$

The former expression corresponds in general to the homogeneous field equations. This identity is in fact equivalent to the equations obtained from the usual approaches (Dyson, 1990; Lee, 1990; Hughes, 1992; Tanimura, 1992; Bracken, 1998). Now, since  $\partial^\mu \partial^\nu F_{\mu\nu} = 0$  holds, there must exist a conserved current given by

$$j_\mu \equiv \partial^\nu F_{\mu\nu} \quad (22)$$

which as usual can be identified as the source of the fields (Jackson, 1975). Therefore, the last equation corresponds to the nonhomogeneous field equation, which is not obtained in the original scheme by Feynman (Dyson, 1990), nor in the extended versions of the proof (Lee, 1990; Hughes, 1992; Tanimura, 1992; Bracken, 1998). This is the most relevant equation, for it defines the dynamics of the fields (Jackson, 1975). Now, starting from (18), which defines  $F_{\mu\nu}$ , we note that the relation

$$F_{\mu\nu} \dot{x}^\nu = \left\{ m\dot{x}_\mu, \frac{1}{2} m\dot{x}_\nu \dot{x}^\nu \right\} \quad (23)$$

holds, which suggests including the Hamiltonian of the system (Goldstein, 1980)

$$H = \frac{1}{2m} (\pi - A)^2 = \frac{1}{2} m\dot{x}_\nu \dot{x}^\nu \quad (24)$$

and obtaining a generalized Lorentz law

$$F_{\mu\nu} \dot{x}^\nu = m\ddot{x}_\mu \quad (25)$$

In summary, starting only from the minimal coupling rule (12), we were able to obtain the tensor of the interaction fields  $F_{\mu\nu}$  as well as the equation of motion of a test particle (25) and the analog to the field equations [Eqs. (21) and (22)]. In the next two sections we will apply explicitly this method to both the Abelian and non-Abelian cases.

#### 4. THE ABELIAN CASE: ELECTROMAGNETIC FIELD

The electromagnetic case is the simplest one. Let us take the following restriction on the fundamental hypothesis [given by Eq. (12)]:

$$A_\mu = A_\mu(x) \quad (26)$$

i.e.,  $\tilde{\partial}_\mu A_\nu = 0$ , which means, for the present case, that (15) reduces to  $m\{x_\mu, x_\nu\} = \eta_{\mu\nu}$  (the usual starting point of Feynman's proof. Note also that  $\{A_\mu, A_\nu\} = 0$  holds. Therefore, (19) allows us to define the electromagnetic tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (27)$$

which satisfies the Bianchi identity [obtained from (20)]

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \quad (28)$$

which corresponds to the homogeneous Maxwell equations. The equations with sources can be obtained from  $j_\mu \equiv \partial^\nu F_{\mu\nu}$  in (22). Explicitly, as usual,  $E_i = F_{0i}$  and  $H_i = \tilde{F}_{0i}$ , where  $\tilde{F}_{\mu\nu} = 1/2\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$  is the dual tensor, and  $i = 1, 2, 3$ . Consequently, the electromagnetic fields are defined by  $\mathbf{E} = -\partial_0\mathbf{A} - \nabla A_0$  and  $\mathbf{H} = \nabla \times \mathbf{A}$ .

Note also that the Lorentz law is clearly (25). We have used explicitly the Hamiltonian of the test particle in order to obtain this equation of motion, but this expression can be obtained by integrating (18) with respect to  $\pi_\mu$ .

In summary, we have obtained the complete set of the Maxwell equations and the Lorentz law for the test particle (without assuming it from the beginning) just starting from the hypothesis (26). It is important to emphasize this property, of the present approach, because it is not shared by the Feynman's proof (Dyson, 1990) nor its direct extensions (Lee, 1990; Hughes, 1992; Tanimura, 1992; Bracken, 1998). In others words, the four Maxwell equations (with sources and without magnetic monopole terms) emerge in a natural way if a coupling of the form (26) is assumed, which means that  $A_\mu$  does not depend on the velocity of the test particle. Therefore, all the dynamic information of the system is contained in the minimal coupling rule, as we claimed.

## 5. THE NON-ABELIAN GAUGE FIELDS

As we noted above, the general form of the field tensor [Eq. (19)] is that of the non-Abelian case. It suggests that the classical non-Abelian gauge field equations could be obtained from the minimal coupling rule through a special condition over  $\{A_\mu, A_\nu\}$ . Basically we have to note that in the classical approach the non-Abelian fields may be treated by introducing new internal degrees of freedom, such as the isospin, in such a way that the Hamiltonian would depend on some other non spacetime variables. Some steps in this direction were made by Lee (1990) and Tanimura (1992) and recently discussed in a relativistic context by Bracken (1998).

Let us consider as the canonical coordinates of the test particle those which belong to a  $(d + n)$ -dimensional space, where  $d$  is the spacetime dimension and  $n$  is the internal space dimension (for instance, isospin), which is necessary to “balance” the momentum due to the external interaction. We use the following notations and conventions:  $\Omega, \Lambda = 0, 1, \dots, d, \dots, d + n$ ;  $\alpha, \mu, \nu = 0, 1, \dots, d$ ; and  $a, b, c = d + 1, \dots, d + n$ ; in such a way that all the results obtained in the Section 3 hold on the indices  $\Omega, \Lambda$ .

Next, let us assume the dependence on the canonical coordinates of  $A_\Omega$  is such that it is separable, and can be written in the form

$$A_\Omega = A_\Omega(x_\Lambda, \pi_a) = A_{\Omega a}(x_\mu) I^a(x_b, \pi_b) \quad (29)$$

which means it does not depend on the canonical momentum associated to the spacetime coordinates. Also let us assume that  $I^a$  satisfies

$$\{I^a, I^b\} = -f^{ab}{}_c I^c \quad (30)$$

where  $f^{ab}{}_c$  are the structure constants corresponding to the Lie algebra of the Lie group locally generated by the quantum operators associated to the functions  $I^a$ .

Note that due to the separation of the coordinates, the Poisson brackets can be written in the form  $\{A, B\} = \{A, B\}_{\text{esp}} + \{A, B\}_{\text{int}}$  where “esp” and “int” mean spacetime and internal space, respectively. Under this consideration and taking (29) into account, we will have  $\{A_\Omega, A_\Lambda\} = A_{\Omega a} A_{\Lambda b} \{I^a, I^b\}$ , for the spacetime part of the bracket vanishes. Hence, expressing the equations only in the spacetime coordinate sector, we find that (19) can be written as  $F_{\mu\nu} = (\partial_\mu A_{\nu c} - \partial_\nu A_{\mu c} - A_{\mu a} A_{\nu b} f^{ab}{}_c) I^c$ , from which it is natural to interpret the term between brackets in the former equation as the *Yang–Mills field tensor* given by

$$F_{\mu\nu c} \equiv \partial_\mu A_{\nu c} - \partial_\nu A_{\mu c} - A_{\mu a} A_{\nu b} f^{ab}{}_c \quad (31)$$

Note that because  $A_\Omega$  does not depend on  $\pi_\nu$ , the equation of motion (25) can also be obtained by integrating (18), acquiring the form

$$m\ddot{x}_\mu = F_{\mu\nu a} I^a \dot{x}^\nu + G_{\mu a} I^a \quad (32)$$

which is the *first Wong equation for the non-Abelian gauge fields*. As we shall see, the last term in the equation above, absent from (25), may be identified as a gauge term (see last part of this section).

On the other hand, the covariant derivative can be defined as

$$(D_\alpha F_{\mu\nu})_c \equiv \partial_\alpha F_{\mu\nu c} - f^{ba}{}_c A_{ab} F_{\mu\nu a} \quad (33)$$

implying the Bianchi identity

$$(D_\alpha F_{\mu\nu})_c + (D_\mu F_{\nu\alpha})_c + (D_\nu F_{\alpha\mu})_c = 0 \quad (34)$$



Also from  $\dot{I}^a = \{I^a, H\} = m\dot{x}^\mu\{I^a, \dot{x}_\mu\}$  and

$$m\{I^a, \dot{x}_\mu\} = A_{\mu b}F^{ab}{}_c \quad (35)$$

we get the *second Wong equation*

$$\dot{I}^a - f^{ab}{}_c A_{\nu b}\dot{x}^\nu I^c = 0 \quad (36)$$

or equivalently  $\{x_\mu, (\dot{I}^a - f^{ab}{}_c A_{\nu b}\dot{x}^\nu I^c)\} = 0$ . Finally, from (35) we obtain the usual expression for functions of the type  $\phi_a(x)$ :

$$m\{\dot{x}_\mu, \phi_a(x)I^a\} = -\{\partial_\mu\phi_a - f^{bc}{}_a A_{\mu b}\phi_c\}I^a = -(D_\mu\phi)_a I^a \quad (37)$$

In particular,  $G_\mu \equiv G_{\mu a}I^a$  satisfies  $m\{\dot{x}_\mu, G_\nu\} = -(D_\mu G_\nu)_a I^a$ , which together with the two Wong equations implies  $G_\mu$  is a gauge term because

$$(D_\mu G_\nu)_a - (D_\nu G_\mu)_a = 0 \quad (38)$$

holds.

## 6. CONCLUDING REMARKS

We summarize the above results as follows. Even though Feynman's proof fails because it provides no new physics, the proof is successful because it reduces the laws of the gauge interactions in the sense that they can be obtained from only the minimal coupling postulate. This fact is not only an economical choice, but it has a deeper meaning which for the present analysis signifies that the fundamental dynamic equations, the field equations, and the motion equation of the test particle (the Lorentz law) just come from the minimal coupling rule between the potential and the linear momentum. However, this fundamental fact is unclear (and missed in the discussions) of the approaches that start from postulating the quantum commutators or the equivalent Poisson brackets.

On the other hand, it is important to emphasize some aspects of the present approach. First, we are in the framework of relativistic classical mechanics. Second, a quantum point of view has not been adopted, so the relationship with quantization schemes should be analyzed carefully. In particular, the relationship with the Dirac method (Dirac, 1964) would be interesting to study because of the implicit dependence of the gauge fields on the momenta [see (12)]. We are aware that the present approach might be introducing second-class constraints in a quantum analysis (as in QED). If this were the case, a more careful treatment should be given if a quantization based on the present approach is considered. These conjectures are beyond the scope of the present paper, but they should be clarified.

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